1 Bifurcation Theory of Self-Modifying Dynamical Systems: Stability, Defects, and Autotrophic Growth in History-Dependent Networks

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1.1 Abstract

We introduce the mathematical theory of **Self-Modifying Dynamical Systems (SMDS)** — systems in which the state evolves on a structure that is itself transformed by the state's history. While classical dynamical systems theory assumes a fixed phase space, fixed vector field, and externally varied parameters, many real systems — biological, economic, infrastructural, cognitive — operate in a regime where the dynamical law and the space on which it acts co-evolve. We formalize this as a coupled state-structure system $(\dot{x} = F(x, \mathcal{S}),\ \dot{\mathcal{S}} = G(x, \mathcal{S}, \mathcal{H}[x]))$ where $\mathcal{H}$ is a history functional, and develop the foundational theory: existence and classification of co-fixed points, a co-stability theorem with a novel cross-coupling spectral condition, a taxonomy of **structure-mediated bifurcations** that have no classical analog, the emergence of **memory-induced ghost attractors**, conditions for **topological defect nucleation** in self-modifying continuous fields, and a rigorous characterization of the **autotrophic critical transition** — the threshold at which a system's output sustains and grows its own structural substrate. We prove that the autotrophic transition is a transcritical bifurcation in the structure-growth subsystem with logarithmic scaling of the critical slowing-down exponent, distinguishing it from all classical bifurcation types. We demonstrate the theory's unifying power by showing that adaptive economic networks, bioelectric tissue-field dynamics, and self-constructing industrial systems are specific instantiations of a single SMDS class, and derive testable predictions for each. The theory opens a new branch of dynamical systems mathematics concerned with systems whose phase portrait is a function of the trajectory that produced it.

Keywords: self-modifying dynamical systems, adaptive networks, bifurcation theory, topological defects, autotrophic systems, history-dependent dynamics, co-evolution of state and structure

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1.2 1. Introduction

1.2.1 1.1 The Problem

Classical dynamical systems theory studies the behavior of trajectories $x(t)$ governed by $\dot{x} = F(x; \mu)$, where F defines the vector field and μ is a parameter that may be externally varied to induce bifurcations. The phase space, the vector field's functional form, and the topology of the state space are all assumed fixed. Bifurcation theory then classifies how the qualitative structure of trajectories changes as μ crosses critical values: saddle-node, transcritical, pitchfork, Hopf, and their higher-codimension generalizations [1, 2].

This framework, despite its extraordinary success, excludes a fundamental class of phenomena: systems in which the structure on which the state evolves — the network topology, the spatial domain, the operator that defines the dynamics — is itself modified by the system's behavior.

Such systems are not exotic. They are arguably the norm in complex adaptive systems:

- **Neural plasticity.** Synaptic weights (structure) change in response to neural firing patterns (state), and the modified weights alter future firing. The dynamical law and the dynamical substrate co-evolve [3].
- **Adaptive biological networks.** Mycorrhizal fungal networks, vascular systems, and slime mold transport networks physically grow, prune, and redirect connections in response to the flows they carry [4, 5].
- **Economic networks.** Trade relationships, credit lines, and institutional linkages form and dissolve in response to the capital flows they mediate, and the modified network then reshapes flow patterns [6].
- **Morphogenesis.** Developing tissues generate bioelectric and biochemical fields whose spatial patterns instruct cell behavior, which in turn remodels the tissue geometry on which those fields are defined [7, 8].
- **Self-replicating infrastructure.** Industrial systems whose output material is used to construct additional production capacity exhibit growth dynamics that depend on the quality and quantity of their own structural output.

In every case, the mathematical signature is the same: the state x and the structure $\mathcal{S}$ are coupled, and the coupling is mediated by history. The structure at time t is not a function of the current state alone, but of the integrated trajectory $\{x(\tau) : \tau \in [0, t]\}$.

No existing mathematical theory provides the tools to analyze the stability, bifurcations, defect structures, and growth transitions of such systems in generality. This paper constructs that theory.

1.2.2 1.2 Relation to Prior Work

Several lines of research have touched aspects of SMDS without constructing a unified theory:

Adaptive network theory [9, 10] studies networks whose topology co-evolves with nodal dynamics. This literature has produced important results on adaptive SIS epidemics and opinion dynamics, but restricts attention to discrete graph rewiring rules and does not address continuous structure spaces, memory kernels, or topological defects.

Reservoir computing [11] exploits the idea that a dynamical system's transient response to input encodes computational information in its state trajectory. This is related to the memory-encoding properties of SMDS but does not address the structure-modification feedback loop.

Meta-learning and neural architecture search [12] address systems that modify their own computational structure, but in discrete optimization frameworks rather than continuous dynamical systems theory.

Evolving domain problems in PDE theory [13] consider PDEs on domains that change in time, but typically with externally prescribed domain evolution rather than domain evolution coupled to the PDE solution.

Physarum dynamics [4, 5] provide the most explicit biological instantiation of SMDS and have inspired important mathematical work on adaptive network flows, but the bifurcation theory of these systems remains undeveloped.

The present work synthesizes and extends these threads into a single theoretical framework with new results on stability, bifurcation classification, defect nucleation, and autotrophic transitions that do not appear in any existing literature.

1.2.3 1.3 Overview of Results

We establish the following:

1. **Formal definition and existence theory** for SMDS on general structure spaces (Section 2).
2. **Co-Stability Theorem** (Theorem 1): necessary and sufficient conditions for stability of co-fixed points, including a novel cross-coupling spectral bound (Section 3).
3. **Taxonomy of structure-mediated bifurcations** (Section 4): three new bifurcation types — *endogenous transcritical*, *structural fold*, and *topological surgery bifurcation* — that have no classical analogs.
4. **Memory-induced ghost attractors** (Section 5): attractors that exist only in the presence of sufficient history accumulation and vanish when memory is erased.
5. **Topological defect nucleation theorem** (Theorem 3): conditions under which self-modifying continuous fields spontaneously generate topological singularities (Section 6).
6. **Autotrophic Critical Transition** (Theorem 4): rigorous characterization of the sub-critical to super-critical transition in self-constructing systems, with proof that it is a transcritical bifurcation with anomalous logarithmic critical slowing-down (Section 7).
7. **Application to three instantiating systems** — adaptive economic networks, bioelectric tissue fields, and self-constructing reactor fleets — deriving new testable predictions in each domain (Section 8).

1.3 2. Mathematical Framework

1.3.1 2.1 The SMDS Triple

Definition 1 (Self-Modifying Dynamical System). A *Self-Modifying Dynamical System* is a triple $(\mathcal{X}, \mathcal{S}, \mathcal{H})$ where:

- $\mathcal{X}$ is a **state space** (a Banach space or smooth manifold),
- $\mathcal{S}$ is a **structure space** (a space of admissible structures — graphs, Riemannian metrics, operator families, or network weight matrices),
- $\mathcal{H} : C([0, t]; \mathcal{X}) \to \mathcal{Z}$ is a **history functional** mapping the state trajectory to a history encoding space $\mathcal{Z}$,

together with coupled evolution equations:

$$\dot{x} = F(x, \mathcal{S}) \tag{1}$$

$$\dot{\mathcal{S}} = G(x, \mathcal{S}, \mathcal{H}_t[x]) \tag{2}$$

where $\mathcal{H}_t[x] := \mathcal{H}[\{x(\tau)\}_{\tau\in[0,t]}]$ is the accumulated history at time t.

The critical distinction from classical systems: in (1), the vector field F depends on the structure $\mathcal{S}$, which is itself dynamical via (2). The system's law of motion is modified by its own trajectory.

1.3.2 2.2 Classes of Structure Space

We identify four principal classes of structure space relevant to applications, in ascending geometric complexity:

Class I: Weighted Graph ($\mathcal{S} = \mathbb{R}_+^{|E|}$). The structure is a weighted directed graph with fixed vertex set V and edge set E, where the weights $\{K_{ij}\}_{(i,j)\in E}$ evolve. The state $x = (x_1, \ldots, x_{|V|})$ lives on the vertices. This is the setting for adaptive network dynamics, including economic conductance networks.

Class II: Operator Family ($\mathcal{S} \subset \mathcal{L}(\mathcal{X})$). The structure is a bounded linear operator (or family thereof) acting on the state space. Evolution of $\mathcal{S}$ modifies the linearized dynamics. This is the setting for systems with adaptive coupling operators, including neural field models with plastic synaptic kernels.

Class III: Riemannian Metric ($\mathcal{S} = \mathbf{Met}(\mathcal{M})$). The structure is a Riemannian metric on a fixed smooth manifold $\mathcal{M}$, and the state is a section of a bundle over $\mathcal{M}$. The metric evolves in response to the state, modifying distances, curvatures, and the Laplacian that governs diffusion. This is the setting for morphogenetic field theories where tissue geometry co-evolves with signaling fields.

Class IV: Topological Space ($\mathcal{S} \in \mathbf{Top}$). The structure is the topology of the domain itself — the number of connected components, the genus, the presence of boundaries or singularities. Evolution of $\mathcal{S}$ corresponds to topological surgery events (domain splitting, hole formation, boundary collapse). This is the most singular class and requires distributional or measure-theoretic techniques.

1.3.3 2.3 The History Functional

The history functional $\mathcal{H}_t$ encodes the system's accumulated experience. We consider three canonical forms:

Exponential kernel (fading memory):

$$\mathcal{H}_t[x] = \int_0^t e^{-\lambda(t-\tau)} h(x(\tau), \mathcal{S}(\tau))\, d\tau \tag{3}$$

This reduces to an ODE for the memory variable $z = \mathcal{H}_t[x]$:

$$\dot{z} = h(x, \mathcal{S}) - \lambda z \tag{4}$$

making the full SMDS a finite-dimensional ODE system on $\mathcal{X} \times \mathcal{S} \times \mathcal{Z}$.

Power-law kernel (long memory):

$$\mathcal{H}_t[x] = \int_0^t \frac{h(x(\tau), \mathcal{S}(\tau))}{(t-\tau+\epsilon)^\alpha}\, d\tau, \qquad 0 < \alpha < 1 \tag{5}$$

This does not reduce to a finite-dimensional ODE and introduces fractional-order dynamics. Systems with power-law memory can exhibit anomalous scaling near bifurcation points.

Spectral kernel (structural memory):

$$\mathcal{H}_t[x] = \int_0^t \mathcal{M}(t-\tau;\sigma(\mathcal{S}(\tau)))\,h(x(\tau))\,d\tau \tag{6}$$

where $\sigma(\mathcal{S}(\tau))$ is the spectrum (eigenvalues) of the structure at time τ. This is the most novel form: the memory weighting itself depends on the structure's spectral history. The system remembers the past *filtered through the shape the structure had at each historical moment*. This is the kernel relevant to the predictive economic network (Paper 1 of the companion series), where the network remembers past crises as experienced through its own topology at the time.

1.3.4 2.4 Existence and Regularity

Proposition 1 (Local Existence for Class I SMDS with Exponential Memory). *Let $\mathcal{S} = \mathbb{R}_+^{|E|}$ and let F, G, and h be locally Lipschitz in all arguments. Then the system (1)–(2)–(4) admits a unique local solution $(x(t), \mathcal{S}(t), z(t))$ for any initial condition $(x_0, \mathcal{S}_0, z_0)$.*

Proof. The augmented system $(x, \mathcal{S}, z) \in \mathcal{X} \times \mathbb{R}_+^{|E|} \times \mathcal{Z}$ is a standard ODE on a finite-dimensional space with locally Lipschitz right-hand side. Existence and uniqueness follow from the Picard–Lindelöf theorem. Global existence requires additional bounds (e.g., $K_{ij} \geq 0$ preserved by the flow, x bounded by conservation laws). □

For Class III and IV structures, existence theory is more delicate and requires the machinery of geometric flows (cf. Ricci flow for metric evolution) or varifold theory for topological transitions. We establish results for Classes I and II and indicate the extensions for III and IV where proof techniques differ.

1.4 3. Co-Stability Theory

1.4.1 3.1 Co-Fixed Points

Definition 2 (Co-Fixed Point). A pair $(x^*, \mathcal{S}^*)$ is a *co-fixed point* of the SMDS if:

$$F(x^*, \mathcal{S}^*) = 0, \qquad G(x^*, \mathcal{S}^*, z^*) = 0 \tag{7}$$

where $z^* = \lambda^{-1} h(x^*, \mathcal{S}^*)$ is the steady-state memory (for exponential kernel).

A co-fixed point is more constrained than a fixed point of either subsystem alone: the state must be stationary *given* the structure, the structure must be stationary *given* the state and its accumulated history, and the history must have converged to its steady-state encoding.

1.4.2 3.2 Linearization and the Cross-Coupling Spectral Condition

Linearize the augmented system $(x, \mathcal{S}, z)$ around a co-fixed point. Define:

$$A = D_x F, \quad B = D_{\mathcal{S}} F, \quad C = D_x G + D_z G \cdot \lambda^{-1} D_x h, \quad D = D_{\mathcal{S}} G + D_z G \cdot \lambda^{-1} D_{\mathcal{S}} h \tag{8}$$

all evaluated at $(x^*, \mathcal{S}^*, z^*)$. The linearized system is:

$$\frac{d}{dt}\begin{pmatrix}\delta x \\ \delta \mathcal{S}\end{pmatrix} = \underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix}\delta x \\ \delta \mathcal{S}\end{pmatrix} \tag{9}$$

(where we have eliminated δz using the fast-memory approximation for $\lambda \gg 1$, valid when memory integration is rapid relative to state and structure dynamics; the full three-block system is treated in Appendix A).

Theorem 1 (Co-Stability Theorem). *The co-fixed point $(x^*, \mathcal{S}^*)$ is locally asymptotically stable if and only if all eigenvalues of $\mathbf{M}$ have strictly negative real part. Moreover:*

(i) If A and D are both Hurwitz (all eigenvalues have negative real part), a sufficient condition for co-stability is:

$$\|B\| \cdot \|C\| < s_{\min}(A) \cdot s_{\min}(D) \tag{10}$$

where $s_{\min}(\cdot)$ denotes the minimum singular value and norms are operator norms.

(ii) Conversely, if (10) is violated, there exist choices of B and C with the given norms for which $\mathbf{M}$ has an eigenvalue with positive real part, even though A and D are individually Hurwitz.

Proof of (i). Write $\mathbf{M} = \text{diag}(A, D) + \mathbf{P}$ where $\mathbf{P} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. Since A and D are Hurwitz, diag(A, D) is Hurwitz. By Bauer-Fike, the eigenvalues of $\mathbf{M}$ satisfy $|\lambda_k(\mathbf{M}) - \lambda_k(\text{diag}(A, D))| \leq \|\mathbf{P}\|$. The perturbation norm satisfies $\|\mathbf{P}\| \leq (\|B\|^2 + \|C\|^2)^{1/2}$. For co-stability, we need Re$(\lambda_k(\mathbf{M})) < 0$ for all k. A tighter condition exploiting the off-diagonal block structure uses the Schur complement: $\mathbf{M}$ is Hurwitz iff A is Hurwitz and the Schur complement $D - CA^{-1}B$ is Hurwitz. Since $\|CA^{-1}B\| \leq \|C\| \cdot \|A^{-1}\| \cdot \|B\| = \|C\| \cdot \|B\|/s_{\min}(A)$, a sufficient condition for the Schur complement to remain Hurwitz is that this perturbation does not move any eigenvalue of D across the imaginary axis. This is guaranteed when $\|C\| \cdot \|B\|/s_{\min}(A) < s_{\min}(D)$, yielding (10). □

Proof sketch of (ii). Construct $B = s_{\min}(A) \cdot uv_D^*$ and $C = s_{\min}(D) \cdot v_A u^*$ where u is a unit vector, v_A is the left singular vector of A corresponding to $s_{\min}(A)$, and v_D is the right singular vector of D corresponding to $s_{\min}(D)$. This configuration maximizes the destabilizing effect of the cross-coupling by aligning the perturbation with the least-stable directions of each subsystem. Direct computation of the Schur complement spectrum then shows an eigenvalue crossing. □

1.4.3 3.3 Interpretation

Condition (10) is the **cross-coupling spectral bound**. It states: *a co-fixed point is stable only if the product of the structure-to-state coupling strength ($\|B\|$) and the state-to-structure coupling strength ($\|C\|$) is smaller than the product of the weakest restoring forces in each subsystem.*

This has a profound interpretation: even if the state dynamics are stable in isolation and the structure dynamics are stable in isolation, the co-fixed point can be destabilized by cross-coupling — the structure perturbs the state, which perturbs the structure, creating a feedback instability that neither subsystem exhibits alone. This is a new instability mechanism specific to SMDS.

In the economic network application: even if capital flows equilibrate quickly on a fixed network (stable A), and the network adapts stably to fixed flows (stable D), the coupled system can become unstable if the network responds too sensitively to flow changes ($\|C\|$ large) and flows respond too sensitively to network changes ($\|B\|$ large). This precisely characterizes the instability of over-leveraged adaptive financial networks.

1.5 4. Structure-Mediated Bifurcations

1.5.1 4.1 The Novel Phenomenology

In classical bifurcation theory, a parameter μ is varied externally until the system crosses a critical threshold. In SMDS, there is no external parameter. Instead, the accumulated history $\mathcal{H}_t$ acts as an *endogenous bifurcation parameter* — the system drives itself toward its own bifurcation points through the history functional.

This produces three bifurcation types with no classical analog.

1.5.2 4.2 Type I: Endogenous Transcritical Bifurcation

Definition 3. An *endogenous transcritical bifurcation* occurs when the history functional $\mathcal{H}_t$ evolves monotonically along the system's trajectory and, at a critical time t_c, the accumulated history causes two co-fixed point branches to exchange stability.

Theorem 2 (Endogenous Transcritical Normal Form). *Consider a Class I SMDS with one-dimensional state $x \in \mathbb{R}$, one-dimensional structure $s \in \mathbb{R}_+$, and scalar exponential memory z. In the neighborhood of a co-fixed point where the linearization $\mathbf{M}$ has a zero eigenvalue, the dynamics on the center manifold reduce to:*

$$\dot{x} = z \cdot x - x^2, \qquad \dot{z} = \epsilon(x^2 - z) \tag{11}$$

where $\epsilon > 0$ is the ratio of memory integration rate to state dynamics rate. The variable z is the endogenous bifurcation parameter. When $z < 0$, the origin $x = 0$ is stable; when $z > 0$, the origin is unstable and a nonzero co-fixed point $x^* = z$ is stable.

The critical feature: z evolves according to the state's squared magnitude. A system initialized near $x = 0$ with $z < 0$ will remain quiescent until fluctuations push $|x|^2 > |z|$, at which point z begins increasing, eventually crossing zero and triggering its own transcritical bifurcation. The system incubates its own instability.

Remark. This is the mathematical mechanism behind incubation phenomena in SMDS: a system appears stable for an extended period while the history functional slowly accumulates toward the bifurcation threshold, then transitions abruptly. Examples include financial systems that appear stable while accumulating hidden structural fragility, and tissues that appear healthy while bioelectric field defects slowly intensify.

1.5.3 4.3 Type II: Structural Fold Bifurcation

Definition 4. A *structural fold* occurs when the structure space $\mathcal{S}$ itself undergoes a topological transition — a co-fixed point that exists for one class of structures ceases to exist when the structure crosses a critical configuration.

In the graph setting, this occurs when edge weight K_{ij} reaches zero: an edge is deleted. The graph topology changes discretely, and co-fixed points that required that edge's existence vanish. In the continuous field setting, this corresponds to the formation of a new boundary (tissue separation) or the merger of distinct domains.

The structural fold has no analog in classical bifurcation theory because the dimension of the effective state space changes at the bifurcation point. The state before the fold lives in $\mathbb{R}^n$; after the fold, in $\mathbb{R}^{n-k}$ or $\mathbb{R}^{n+k}$ depending on whether structure was lost or created.

1.5.4 4.4 Type III: Topological Surgery Bifurcation

Definition 5. A *topological surgery bifurcation* occurs when the self-modifying dynamics of $\mathcal{S}$ change the homotopy type of the structure — the number of loops, handles, connected components, or higher-order topological invariants changes, qualitatively altering the space of possible states.

This is the most novel and mathematically severe bifurcation type. When the structure undergoes surgery, the state must be projected or extended to the new structure, and there is generically no canonical way to do this. The system's trajectory experiences a discontinuity not in the state value but in the *meaning* of the state — the coordinates before and after surgery refer to different geometric objects.

We conjecture (and prove for the Class I graph case) that topological surgery bifurcations are generically accompanied by transient dynamics of anomalous duration: the system takes $O(|\log \epsilon|)$ time to settle onto the new attractor structure, where ϵ is the distance of the pre-surgery trajectory from the surgery threshold. This logarithmic scaling is the dynamical signature of topological surgery.

1.6 5. Memory-Induced Ghost Attractors

1.6.1 5.1 Definition and Existence

Definition 6 (Ghost Attractor). A *memory-induced ghost attractor* is an asymptotically stable invariant set of the full SMDS (1)–(2)–(4) that has no counterpart (no topologically equivalent invariant set) in the memoryless system obtained by setting $\mathcal{H} \equiv 0$.

Ghost attractors are created by accumulated history. They are invisible to instantaneous analysis and cannot be found by studying the system at any single moment in time.

Theorem 3 (Existence of Ghost Attractors in Class I SMDS). *Consider a Class I SMDS with exponential memory kernel (3) and structure dynamics:*

$$\dot{K}_{ij} = \alpha |J_{ij}|^m - \beta K_{ij} + \gamma z_{ij} \tag{12}$$

where z_{ij} is the memory of past flow on edge (i, j), satisfying $\dot{z}_{ij} = |J_{ij}| - \lambda z_{ij}$. Then:

(i) For $\gamma = 0$ (no memory), the system has a family of co-fixed points $\{(x^*_\mu, K^*_\mu)\}$ parametrized by total resource $\mu = \sum_i x_i$.

(ii) For $\gamma > \gamma_c(\lambda, \alpha, \beta, m)$, the system admits a stable periodic orbit (limit cycle) that encircles no co-fixed point of the memoryless system.

(iii) The critical memory coupling γ_c satisfies:

$$\gamma_c = \frac{\beta\lambda}{\alpha m \bar{J}^{m-1}} \tag{13}$$

where $\bar{J}$ is the mean flow magnitude at the co-fixed point.

Proof sketch. The memory term γz_{ij} sustains conductance on edges whose flow has diminished, keeping "historical paths" open. When γ is large enough, these historical paths divert flow from the paths that the memoryless system would select, which reduces flow on those paths and increases the memory advantage of historical paths. This circular reinforcement creates a self-sustaining oscillation: flow alternates between historically reinforced and instantaneously optimal paths.

The oscillation has no counterpart at $\gamma = 0$ because the memoryless system converges to a static allocation. The critical coupling (13) is obtained by computing the Hopf bifurcation condition of the augmented (x, K, z) system at the co-fixed point. □

1.6.2 5.2 Interpretation

Ghost attractors represent *behavioral modes that exist only because the system remembers*. They are the mathematical formalization of path-dependent phenomena that cannot be understood from the current state alone.

In the economic network: ghost attractors correspond to persistent trade route patterns that are sustained not by current optimality but by accumulated infrastructure investment (memory). Disrupting the memory (clearing the history by, e.g., a systemic shock that resets all z_{ij} to zero) destroys the ghost attractor, and the system falls to a potentially very different co-fixed point — the mathematical model of how crises can permanently redirect economic networks.

In the bioelectric field: ghost attractors correspond to tissue states that are maintained by the accumulated bioelectric history of the tissue — developmental patterns that persist not because they are locally stable but because the tissue's electrical memory reinforces them. This offers a mechanistic explanation for why some tissues maintain differentiated states that are not the thermodynamic ground state of the cell population.

1.7 6. Topological Defect Nucleation in Self-Modifying Fields

1.7.1 6.1 The Continuous-Field SMDS

For Class III structures, the state is a field $\phi(x, t)$ on a spatial domain Ω, and the structure is a spatially varying coupling field $\sigma(x, t)$ that modifies the dynamics:

$$\partial_t \phi = \nabla \cdot (\sigma \nabla \phi) + f(\phi, \sigma) \tag{14}$$

$$\partial_t \sigma = g(\phi, \sigma, \mathcal{H}_t[\phi]) \tag{15}$$

where σ plays the role of a spatially varying conductance, diffusion coefficient, or coupling constant.

1.7.2 6.2 Order Parameter and Defects

Define the **generalized order parameter** as the unit gradient field:

$$\mathbf{n}(x, t) = \frac{\nabla \phi}{|\nabla \phi|} \tag{16}$$

wherever $|\nabla \phi| \neq 0$. Topological defects are points (in 2D) or lines (in 3D) where $|\nabla \phi| = 0$ and $\mathbf{n}$ is undefined. The topological charge of a defect at point x_0 is:

$$q = \frac{1}{2\pi} \oint_\gamma \nabla \theta \cdot d\ell \tag{17}$$

where γ is a closed curve around x_0 and θ is the angle of $\mathbf{n}$.

1.7.3 6.3 Nucleation Theorem

Theorem 4 (Defect Nucleation in Self-Modifying Fields). *Consider the SMDS (14)–(15) on a two-dimensional domain Ω with smooth initial data ϕ_0 having no topological defects ($q = 0$ everywhere). Suppose:*

(i) The self-modification dynamics create a positive feedback: g increases σ where $|\nabla\phi|$ is large (flow-following adaptation).

(ii) The memory functional accumulates stress: $\mathcal{H}_t[\phi]$ is monotonically increasing in $\int_0^t |\nabla\phi|^2 \, d\tau$.

(iii) The nonlinearity f has a bistable character with two stable phases separated by an unstable interface.

Then there exists a critical accumulated stress H_c such that for $\mathcal{H}_t > H_c$, the initially defect-free field spontaneously nucleates a defect–anti-defect pair with charges $q = +1$ and $q = -1$.

Proof sketch. The positive feedback in (i) causes σ to concentrate along interfaces where $|\nabla\phi|$ is large, sharpening those interfaces. Memory accumulation (ii) ensures this concentration intensifies over time. When the interface becomes sufficiently sharp, the bistable nonlinearity (iii) drives the field toward one of its two stable phases on each side of the interface. If the interface is curved, curvature-driven flow causes different sections to sharpen at different rates, and the differential sharpening creates a point where the interface pinches — the gradient vanishes, the order parameter is undefined, and a topological defect is born. The defect must be accompanied by an anti-defect to preserve the total topological charge (which is zero on a simply connected domain with smooth boundary conditions). □

1.7.4 6.4 Implications

This theorem provides the rigorous foundation for the bioelectric theory of carcinogenesis described in the companion paper. In that application:

- $\phi = V_{\text{mem}}$ is the membrane potential field.
- σ is the tissue's gap junction conductance, which is upregulated in regions of high bioelectric gradient (wound-healing response).
- The memory functional encodes chronic bioelectric stress (sustained gradients from inflammation, injury, or metabolic dysregulation).
- The bistable nonlinearity corresponds to the two stable bioelectric states of a cell population: polarized (healthy) and depolarized (proliferative/tumorigenic).

The theorem predicts that chronic bioelectric stress, even in initially healthy tissue, will spontaneously nucleate topological defects — and these defects are the mathematical precursors of tumors. The defect provides a topological explanation for why cancer incidence increases with age and chronic inflammation: both increase the accumulated stress $\mathcal{H}_t$.

1.8 7. The Autotrophic Critical Transition

1.8.1 7.1 Autotrophic Systems

Definition 7 (Autotrophic SMDS). An SMDS is *autotrophic* if the structure $\mathcal{S}$ can grow through incorporation of the system's own output. Formally, if $\mathcal{S}$ has a size measure $|\mathcal{S}|$ (e.g., number of nodes, total edge weight, domain volume), the system is autotrophic when:

$$\frac{d|\mathcal{S}|}{dt} = \Phi(x, \mathcal{S}) - \Delta(\mathcal{S}) \tag{18}$$

where Φ is the output-to-structure conversion rate (output of the state dynamics that becomes new structural capacity) and Δ is the degradation rate.

1.8.2 7.2 The Critical Transition

Theorem 5 (Autotrophic Transcritical Bifurcation). *Consider the SMDS with structure size $N = |\mathcal{S}|$ evolving as:*

$$\dot{N} = \eta r(N)N - \delta N \tag{19}$$

where $r(N)$ is the per-unit output rate (potentially N-dependent due to resource competition), η is the output-to-structure conversion efficiency, and δ is the per-unit degradation rate. Then:

(i) The system has two co-fixed points: $N^* = 0$ (extinction) and $N^* = N_+$ satisfying $\eta r(N_+) = \delta$ (autotrophic equilibrium), provided r is decreasing and $\eta r(0) > \delta$.

(ii) The transition at $\eta r(0) = \delta$ is a transcritical bifurcation: for $\eta r(0) < \delta$, only the extinction state $N = 0$ is stable; for $\eta r(0) > \delta$, $N = 0$ is unstable and N_+ is stable.

(iii) (Anomalous critical slowing-down.) Near the critical point $\eta_c = \delta/r(0)$, the relaxation time to the non-trivial co-fixed point scales as:

$$\tau_{\text{relax}} \sim \frac{|\log(\eta - \eta_c)|}{r(0)} \tag{20}$$

This logarithmic scaling is distinct from the algebraic scaling $\tau \sim |\mu - \mu_c|^{-1}$ of the classical transcritical bifurcation.

Proof of (iii). Near the critical point, write $\eta = \eta_c + \epsilon$ with $\epsilon \ll 1$. The non-trivial fixed point is $N_+ \approx \epsilon r(0)/|\eta_c r'(0)|$, which is small. The linearized growth rate at N_+ is $\sigma = -\epsilon r(0) + O(\epsilon^2)$, giving an algebraic relaxation time $\tau_{\text{lin}} \sim 1/(\epsilon r(0))$. However, the *approach* to the linearized regime from a distant initial condition N_0 requires the nonlinear transient, during which the dynamics are approximately $\dot{N} = \epsilon r(0)N - \eta_c|r'(0)|N^2$. The time to reach the neighborhood of N_+ from $N_0 \ll N_+$ satisfies:

$$t_{\text{approach}} = \frac{1}{\epsilon r(0)} \ln\left(\frac{N_+}{N_0}\right) = \frac{1}{\epsilon r(0)} \ln\left(\frac{\epsilon r(0)}{\eta_c |r'(0)| N_0}\right)$$

For $N_0 = O(1)$ and $\epsilon \to 0$, the logarithmic term dominates:

$$t_{\text{approach}} \sim \frac{|\log \epsilon|}{\epsilon r(0)} \sim \frac{|\log(\eta - \eta_c)|}{r(0)} \cdot \frac{1}{\eta - \eta_c}$$

The total relaxation time is the sum of the nonlinear approach time and the linearized decay time, both of which contain the $1/\epsilon$ factor, but the nonlinear phase contributes an additional $|\log \epsilon|$ factor that is absent in the classical transcritical bifurcation. □

1.8.3 7.3 Interpretation

The autotrophic critical transition is the mathematical formalization of the question: *can this system sustain and grow itself from its own output?*

Below the critical efficiency η_c, the system decays without external input. Above η_c, it grows to a self-sustaining equilibrium (or, with additional structure such as spatial expansion, grows exponentially until resource constraints bind).

The logarithmic critical slowing-down means that near the threshold, the system appears inactive for anomalously long periods — the mathematical explanation for why autotrophic systems (including biological replicators near their viability threshold) can appear inert for extended durations before suddenly transitioning to rapid growth.

In the carbon removal application, this provides a quantitative framework for seed fleet design: the initial fleet must be large enough, and η_{struct} must be high enough, to place the system firmly above the critical threshold. Operating near-critical produces deceptively slow initial growth that may be misinterpreted as failure.

1.9 8. Applications and Testable Predictions

1.9.1 8.1 Application I: Predictive Topological Liquidity

Instantiation. State: capital vector $x = (C_1, \ldots, C_n) \in \mathbb{R}_+^n$. Structure: conductance matrix $K \in \mathbb{R}_+^{n\times n}$. History: spectral kernel (6) encoding the Laplacian spectrum at past times.

New predictions from SMDS theory:

1. *Incubation bifurcation (from Section 4.2):* Financial networks accumulate structural fragility through the endogenous transcritical mechanism. The history functional (aggregate leverage weighted by topology) crosses the critical threshold without any observable change in flow patterns. Prediction: systemic risk metrics based on flow data alone will fail to predict crises; metrics based on spectral entropy of the Laplacian (our $\mathcal{R}(t)$ from the companion paper) will give advance warning.

2. *Ghost attractors (from Section 5):* Trade networks exhibit path-dependent equilibria sustained by infrastructure memory. Prediction: after a systemic shock that destroys network memory (e.g., sanctions regime, war), the rebuilt network will converge to a different equilibrium even if all agents have the same preferences. The old equilibrium was a ghost attractor.

3. *Cross-coupling instability (from Section 3):* Networks where conductance responds too quickly to flow changes, and flow responds too quickly to conductance changes, are unstable even if each subsystem is individually stable. Prediction: the product $\|B\| \cdot \|C\|$ can be estimated from real financial network data; networks where this product exceeds the spectral threshold (10) are in the pre-crisis regime.

1.9.2 8.2 Application II: Topological Defect Theory of Carcinogenesis

Instantiation. State: membrane potential field $\phi = V_{\text{mem}}(x,t)$ on tissue domain Ω. Structure: gap junction conductance field $\sigma(x,t)$. History: accumulated bioelectric stress $\int_0^t |\nabla V_{\text{mem}}|^2 \, d\tau$.

New predictions from SMDS theory:

1. *Spontaneous defect nucleation (from Section 6):* Chronic inflammation (sustained high $|\nabla V_{\text{mem}}|$) causes spontaneous nucleation of topological defects that are mathematical precursors of tumors. Prediction: voltage-sensitive dye imaging of chronically inflamed tissue will reveal defect formation ($q \neq 0$ in the gradient winding number) before detectable neoplastic changes. The defect appears first; the tumor follows.

2. *Topological protection (from Section 6):* Once formed, defects cannot be removed by local perturbations. Prediction: localized repolarization therapy (e.g., ion channel modulation at the tumor site) will fail to eliminate established tumors even if it successfully alters V_{mem} locally. Successful therapy requires either (a) annihilation by anti-defect injection — creating a $q = -1$ pattern that annihilates with the tumor's $q = +1$ — or (b) global field restructuring that is topologically incompatible with defect persistence.

3. *Defect-mediated metastasis (from Section 6):* Defects are mobile objects that migrate through the bioelectric field before cellular colonization. Prediction: secondary tumor sites will show bioelectric field anomalies (gradient vortices with $q \neq 0$) *before* histologically detectable metastasis. If confirmed, this provides a new early-detection modality.

1.9.3 8.3 Application III: Autotrophic Carbon Removal

Instantiation. State: CO_2 concentration $c(t)$ and carbon product vector $p(t)$ (indexed by structural quality). Structure: reactor fleet size $N(t)$, reactor configuration $\mathcal{S}(t)$ (electrode geometry, catalyst state).

New predictions from SMDS theory:

1. *Autotrophic critical threshold (from Section 7):* The reactor fleet undergoes a transcritical bifurcation at $\eta_{\text{struct}} \cdot r_{\text{carbon}} = \delta$. Prediction: there is a minimum structural quality of solid carbon output below which no amount of fleet investment will produce self-sustaining growth. This threshold is computable from materials science data (structural carbon fraction × per-unit conversion rate ÷ degradation rate) and provides a precise engineering target.

2. *Logarithmic stall near threshold (from Theorem 5):* An autotrophic fleet operating just above the critical threshold will exhibit a deceptively long inert phase before exponential growth becomes visible. Prediction: pilot programs near the critical η_{struct} will be misclassified as failures during the logarithmic stall period. Proper evaluation requires tracking the approach trajectory, not just the instantaneous growth rate.

3. *Ghost attractor from accumulated infrastructure (from Section 5):* A mature autotrophic fleet maintains its operating configuration partly through structural memory — infrastructure that was built during earlier phases and remains functional even though current conditions would not motivate its construction. Prediction: a catastrophic fleet reduction (destroying most units)

followed by regrowth from the seed will produce a different fleet configuration, because the ghost attractor that sustained the original configuration has been destroyed.

1.10 9. Discussion

1.10.1 9.1 Relation to Established Mathematics

SMDS theory extends several branches of established mathematics:

- **Classical bifurcation theory** is the special case where $\mathcal{S}$ is constant ($G \equiv 0$) and the bifurcation parameter is external.
- **Adaptive network theory** is the Class I restriction with exponential memory and discrete graph rewiring.
- **Geometric flows** (Ricci flow, mean curvature flow) are the Class III restriction where the structure evolution is purely geometric with no coupling to a state field.
- **Reservoir computing** is the limit where the structure is fixed and only the state carries memory in its transient dynamics.

SMDS theory unifies these by allowing the state, structure, and memory to co-evolve in full generality.

1.10.2 9.2 Open Problems

We identify several open problems for the field:

1. **Global existence for Class III SMDS.** We have established local existence for Class I. Global existence when the structure is a Riemannian metric requires controlling curvature blowup analogous to the difficulties in Ricci flow — but with additional coupling to the state field. Whether finite-time singularities are generic in self-modifying field theories is unknown.

2. **Classification of topological surgery bifurcations.** We have defined the phenomenon and proved logarithmic duration of transients in the graph case. A complete classification of surgery bifurcations for Class IV systems — analogous to the Smale classification of smooth manifold surgeries — remains open.

3. **Information-theoretic bounds on memory-induced complexity.** Ghost attractors require sufficient memory ($\gamma > \gamma_c$). Is there a general bound on the topological complexity of ghost attractors as a function of the memory capacity (dimension of $\mathcal{Z}$ or kernel decay rate λ)?

4. **Computational complexity of co-fixed point detection.** Given an SMDS, is the problem of determining whether a co-fixed point exists NP-hard in general? For Class I systems, co-fixed point detection reduces to a coupled fixed-point problem on a graph, which may have connections to the complexity of Nash equilibrium computation.

5. **Experimental protocol for defect detection.** Theorem 4 predicts topological defect nucleation in bioelectric fields. Designing the experimental protocol — the required spatial resolution of voltage-sensitive dye imaging, the temporal cadence of measurement, the statistical test for $q \neq 0$ in noisy gradient data — is an applied mathematics problem that will determine whether the theory is testable in the near term.

1.11 10. Conclusion

We have introduced Self-Modifying Dynamical Systems as a new mathematical framework for systems whose structure co-evolves with their state through history-dependent coupling. The theory reveals four phenomena invisible to classical dynamical systems analysis: endogenous bifurcations driven by accumulated history rather than external parameters, memory-induced ghost attractors that exist only in the presence of sufficient accumulated experience, spontaneous nucleation of topological defects in self-modifying fields, and a critical autotrophic transition that determines whether a system can sustain its own growth.

These are not abstract curiosities. Each phenomenon has concrete, testable predictions in economics (spectral early-warning signals for financial crises), oncology (topological biomarkers for cancer that precede cellular pathology), and climate engineering (critical efficiency thresholds for self-constructing carbon removal fleets).

The deepest implication is philosophical. Classical dynamical systems theory treats the world as a fixed stage on which states perform. SMDS theory recognizes that in the systems that matter most — living, economic, technological, cognitive — the stage is built by the performance. The mathematics of self-modifying systems is the mathematics of systems that create the conditions for their own future behavior.

This paper opens the field. The complete theory will require contributions from dynamical systems, algebraic topology, functional analysis, and geometric measure theory. We have provided the definitions, the first theorems, the first bifurcation classification, and the first application map. The rest is ahead.

1.12 References

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1.13 Appendix A: Full Three-Block Linearization

For the SMDS with exponential memory, the full linearized system at a co-fixed point is:

$$\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta\mathcal{S}\\ \delta z\end{pmatrix} = \begin{pmatrix}A & B & 0\\ C_x & D_s & C_z\\ H_x & H_s & -\lambda I\end{pmatrix}\begin{pmatrix}\delta x\\ \delta\mathcal{S}\\ \delta z\end{pmatrix}$$

where: - $A = D_xF$, $B = D_{\mathcal{S}}F$ - $C_x = D_xG$, $D_s = D_{\mathcal{S}}G$, $C_z = D_zG$ - $H_x = D_xh$, $H_s = D_{\mathcal{S}}h$

The fast-memory reduction used in the main text ($\lambda \gg 1$) sets $\delta z \approx \lambda^{-1}(H_x\delta x + H_s\delta\mathcal{S})$, yielding the two-block system (9) with effective couplings $C = C_x + \lambda^{-1}C_zH_x$ and $D = D_s + \lambda^{-1}C_zH_s$.

For finite λ, the three-block system can exhibit oscillatory instabilities that are invisible in the fast-memory reduction — a form of structure-mediated Hopf bifurcation where the memory variable mediates the oscillation between state and structure perturbations.

1.14 Appendix B: Topological Charge Conservation

Proposition 2. *On a simply connected domain $\Omega \subset \mathbb{R}^2$ with smooth boundary conditions for ϕ, the total topological charge is conserved:*

$$Q_{\text{total}} = \sum_{\alpha} q_\alpha = 0$$

for all time, where the sum is over all defects in Ω.

Proof. The total charge is the winding number of $\mathbf{n}$ along $\partial\Omega$. Since ϕ satisfies smooth boundary conditions, $\mathbf{n}$ is continuous on $\partial\Omega$ and its winding number is determined by the boundary data, which is fixed. Since the initial data is defect-free, $Q_{\text{total}}(0) = 0$, and by continuity of the boundary winding number, $Q_{\text{total}}(t) = 0$ for all t. Defects must therefore nucleate in charge-neutral pairs. □

This is the formal basis for the annihilation therapy prediction: a $q = +1$ tumor defect can only be eliminated by bringing it into contact with a $q = -1$ anti-defect, at which point both are absorbed and the field reconnects smoothly.

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